

## Twelfth International Olympiad, 1970

### 1970/1.

Let  $M$  be a point on the side  $AB$  of  $\triangle ABC$ . Let  $r_1, r_2$  and  $r$  be the radii of the inscribed circles of triangles  $AMC, BMC$  and  $ABC$ . Let  $q_1, q_2$  and  $q$  be the radii of the escribed circles of the same triangles that lie in the angle  $ACB$ . Prove that

$$\frac{r_1}{q_1} \cdot \frac{r_2}{q_2} = \frac{r}{q}.$$

### 1970/2.

Let  $a, b$  and  $n$  be integers greater than 1, and let  $a$  and  $b$  be the bases of two number systems.  $A_{n-1}$  and  $A_n$  are numbers in the system with base  $a$ , and  $B_{n-1}$  and  $B_n$  are numbers in the system with base  $b$ ; these are related as follows:

$$\begin{aligned} A_n &= x_n x_{n-1} \cdots x_0, & A_{n-1} &= x_{n-1} x_{n-2} \cdots x_0, \\ B_n &= x_n x_{n-1} \cdots x_0, & B_{n-1} &= x_{n-1} x_{n-2} \cdots x_0, \\ x_n &\neq 0, & x_{n-1} &\neq 0. \end{aligned}$$

Prove:

$$\frac{A_{n-1}}{A_n} < \frac{B_{n-1}}{B_n} \text{ if and only if } a > b.$$

### 1970/3.

The real numbers  $a_0, a_1, \dots, a_n, \dots$  satisfy the condition:

$$1 = a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots$$

The numbers  $b_1, b_2, \dots, b_n, \dots$  are defined by

$$b_n = \sum_{k=1}^n \left(1 - \frac{a_{k-1}}{a_k}\right) \frac{1}{\sqrt{a_k}}.$$

(a) Prove that  $0 \leq b_n < 2$  for all  $n$ .

(b) Given  $c$  with  $0 \leq c < 2$ , prove that there exist numbers  $a_0, a_1, \dots$  with the above properties such that  $b_n > c$  for large enough  $n$ .

### 1970/4.

Find the set of all positive integers  $n$  with the property that the set  $\{n, n+1, n+2, n+3, n+4, n+5\}$  can be partitioned into two sets such that the product of the numbers in one set equals the product of the numbers in the other set.

**1970/5.**

In the tetrahedron  $ABCD$ , angle  $BDC$  is a right angle. Suppose that the foot  $H$  of the perpendicular from  $D$  to the plane  $ABC$  is the intersection of the altitudes of  $\triangle ABC$ . Prove that

$$(AB + BC + CA)^2 \leq 6(AD^2 + BD^2 + CD^2).$$

For what tetrahedra does equality hold?

**1970/6.**

In a plane there are 100 points, no three of which are collinear. Consider all possible triangles having these points as vertices. Prove that no more than 70% of these triangles are acute-angled.