First International Olympiad, 1959

1959/1.

Prove that the fraction $\frac{21n+4}{14n+3}$ is irreducible for every natural number n.

1959/2.

For what real values of x is

$$\sqrt{(x+\sqrt{2x-1})}+\sqrt{(x-\sqrt{2x-1})}=A,$$

given (a) $A=\sqrt{2}$, (b) $A=1$, (c) $A=2$, where only non-negative real numbers are admitted for square roots?

1959/3.

Let a, b, c be real numbers. Consider the quadratic equation in $\cos x$:

$$a\cos^2 x+b\cos x+c=0.$$

Using the numbers a, b, c, form a quadratic equation in $\cos 2x$, whose roots are the same as those of the original equation. Compare the equations in $\cos x$ and $\cos 2x$ for $a=4, b=2, c=-1$.

1959/4.

Construct a right triangle with given hypotenuse c such that the median drawn to the hypotenuse is the geometric mean of the two legs of the triangle.

1959/5.

An arbitrary point M is selected in the interior of the segment AB. The squares $AMCD$ and $MBEF$ are constructed on the same side of AB, with the segments AM and MB as their respective bases. The circles circumscribed about these squares, with centers P and Q, intersect at M and also at another point N. Let N' denote the point of intersection of the straight lines AF and BC.
(a) Prove that the points N and N' coincide.
(b) Prove that the straight lines MN pass through a fixed point S independent of the choice of M.
(c) Find the locus of the midpoints of the segments PQ as M varies between A and B.

1959/6.

Two planes, P and Q, intersect along the line p. The point A is given in the plane P, and the point C in the plane Q; neither of these points lies on the straight line p. Construct an isosceles trapezoid $ABCD$ (with AB parallel to CD) in which a circle can be inscribed, and with vertices B and D lying in the planes P and Q respectively.