

## Problem 1.

**Solution.** As  $a^3b - 1 = b(a^3 + 1) - (b + 1)$  and  $a + 1 \mid a^3 + 1$ , we have  $a + 1 \mid b + 1$ .

As  $b^3a + 1 = a(b^3 - 1) + (a + 1)$  and  $b - 1 \mid b^3 - 1$ , we have  $b - 1 \mid a + 1$ .

So  $b - 1 \mid b + 1$  and hence  $b - 1 \mid 2$ .

- If  $b = 2$ , then  $a + 1 \mid b + 1 = 3$  gives  $a = 2$ . Hence  $(a, b) = (2, 2)$  is the only solution in this case.
- If  $b = 3$ , then  $a + 1 \mid b + 1 = 4$  gives  $a = 1$  or  $a = 3$ . Hence  $(a, b) = (1, 3)$  and  $(3, 3)$  are the only solutions in this case.

To summarize,  $(a, b) = (1, 3)$ ,  $(2, 2)$  and  $(3, 3)$  are the only solutions.



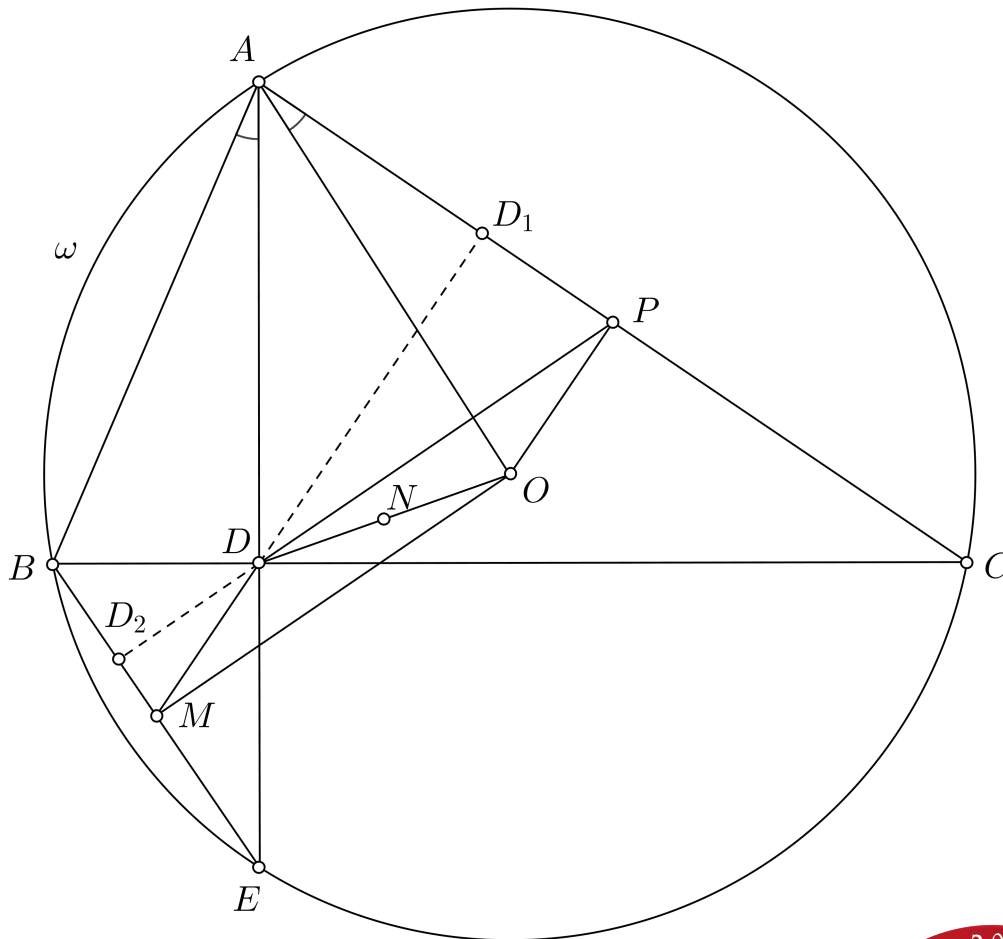
## Problem 2.

**Solution.** We will show that  $MOPD$  is a parallelogram. From this it follows that  $M, N, P$  are collinear.

Since  $\angle BAD = \angle CAO = 90^\circ - \angle ABC$ ,  $D$  is the foot of the perpendicular from  $A$  to side  $BC$ . Since  $M$  is the midpoint of the line segment  $BE$ , we have  $BM = ME = MD$  and hence  $\angle MDE = \angle MED = \angle ACB$ .

Let the line  $MD$  intersect the line  $AC$  at  $D_1$ . Since  $\angle ADD_1 = \angle MDE = \angle ACD$ ,  $MD$  is perpendicular to  $AC$ . On the other hand, since  $O$  is the center of the circumcircle of triangle  $ABC$  and  $P$  is the midpoint of the side  $AC$ ,  $OP$  is perpendicular to  $AC$ . Therefore  $MD$  and  $OP$  are parallel.

Similarly, since  $P$  is the midpoint of the side  $AC$ , we have  $AP = PC = DP$  and hence  $\angle PDC = \angle ACB$ . Let the line  $PD$  intersect the line  $BE$  at  $D_2$ . Since  $\angle BDD_2 = \angle PDC = \angle ACB = \angle BED$ , we conclude that  $PD$  is perpendicular to  $BE$ . Since  $M$  is the midpoint of the line segment  $BE$ ,  $OM$  is perpendicular to  $BE$  and hence  $OM$  and  $PD$  are parallel.



### Problem 3.

**Solution 1.** By the AM-GM Inequality we have:

$$\frac{a+1}{2} + \frac{2}{a+1} \geq 2$$

Therefore

$$a + 2b + \frac{2}{a+1} \geq \frac{a+3}{2} + 2b.$$

and, similarly,

$$b + 2a + \frac{2}{b+1} \geq 2a + \frac{b+3}{2}.$$

On the other hand,

$$(a + 4b + 3)(b + 4a + 3) \geq (\sqrt{ab} + 4\sqrt{ab} + 3)^2 \geq 64$$

by the Cauchy-Schwarz Inequality as  $ab \geq 1$ , and we are done.

**Solution 2.** Since  $ab \geq 1$ , we have  $a + b \geq a + 1/a \geq 2\sqrt{a \cdot (1/a)} = 2$ .

Then

$$\begin{aligned} a + 2b + \frac{2}{a+1} &= b + (a+b) + \frac{2}{a+1} \\ &\geq b + 2 + \frac{2}{a+1} \\ &= \frac{b+1}{2} + \frac{b+1}{2} + 1 + \frac{2}{a+1} \\ &\geq 4\sqrt[4]{\frac{(b+1)^2}{2(a+1)}} \end{aligned}$$

by the AM-GM Inequality. Similarly,

$$b + 2a + \frac{2}{b+1} \geq 4\sqrt[4]{\frac{(a+1)^2}{2(b+1)}}.$$



Now using these and applying the AM-GM Inequality another time we obtain:

$$\begin{aligned} \left(a + 2b + \frac{2}{a+1}\right) \left(b + 2a + \frac{2}{b+1}\right) &\geq 16 \sqrt[4]{\frac{(a+1)(b+1)}{4}} \\ &\geq 16 \sqrt[4]{\frac{(2\sqrt{a})(2\sqrt{b})}{4}} \\ &= 16 \sqrt[8]{ab} \\ &\geq 16 \end{aligned}$$

**Solution 3.** We have

$$\begin{aligned} \left(a + 2b + \frac{2}{a+1}\right) \left(b + 2a + \frac{2}{b+1}\right) &= \left((a+b) + b + \frac{2}{a+1}\right) \left((a+b) + a + \frac{2}{b+1}\right) \\ &\geq \left(a + b + \sqrt{ab} + \frac{2}{\sqrt{(a+1)(b+1)}}\right)^2 \end{aligned}$$

by the Cauchy-Schwarz Inequality.

On the other hand,

$$\frac{2}{\sqrt{(a+1)(b+1)}} \geq \frac{4}{a+b+2}$$

by the AM-GM Inequality and

$$a + b + \sqrt{ab} + \frac{2}{\sqrt{(a+1)(b+1)}} \geq a + b + 1 + \frac{4}{a+b+2} = \frac{(a+b+1)(a+b-2)}{a+b+2} + 4 \geq 4$$

as  $a + b \geq 2\sqrt{ab} \geq 2$ , finishing the proof.



## Problem 4.

**Solution. a.** Yes. Let  $a \leq b \leq c \leq d \leq e$  be the numbers chosen by Alice. As each number appears in a pairwise sum 4 times, by adding all 10 pairwise sums and dividing the result by 4, Bob obtains  $a + b + c + d + e$ . Subtracting the smallest and the largest pairwise sums  $a + b$  and  $d + e$  from this he obtains  $c$ . Subtracting  $c$  from the second largest pairwise sum  $c + e$  he obtains  $e$ . Subtracting  $e$  from the largest pairwise sum  $d + e$  he obtains  $d$ . He can similarly determine  $a$  and  $b$ .

**b.** Yes. Let  $a \leq b \leq c \leq d \leq e \leq f$  be the numbers chosen by Alice. As each number appears in a pairwise sum 5 times, by adding all 15 pairwise sums and dividing the result by 5, Bob obtains  $a + b + c + d + e + f$ . Subtracting the smallest and the largest pairwise sums  $a + b$  and  $e + f$  from this he obtains  $c + d$ .

Subtracting the smallest and the second largest pairwise sums  $a + b$  and  $d + f$  from  $a + b + c + d + e + f$  he obtains  $c + e$ . Similarly he can obtain  $b + d$ . He uses these to obtain  $a + f$  and  $b + e$ .

Now  $a + d$ ,  $a + e$ ,  $b + c$  are the three smallest among the remaining six pairwise sums. If Bob adds these up, subtracts the known sums  $c + d$  and  $b + e$  from the result and divides the difference by 2, he obtains  $a$ . Then he can determine the remaining numbers.

**c.** No. If Alice chooses the eight numbers 1, 5, 7, 9, 12, 14, 16, 20, then Bob cannot be sure to guess these numbers correctly as the eight numbers 2, 4, 6, 10, 11, 15, 17, 19 also give exactly the same 28 pairwise sums as these numbers.

