31^{st} Balkan Mathematical Olympiad

May 2-7 2014
Pleven
Bulgaria

Problems and Solutions

Problem 1. Let x, y and z be positive real numbers such that $xy + yz + zx = 3xyz$. Prove that

$$x^2y + y^2z + z^2x \geq 2(x + y + z) - 3$$

and determine when equality holds.

Solution. The given condition can be rearranged to $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 3$. Using this, we obtain:

$$\begin{aligned} x^2y + y^2z + z^2x - 2(x + y + z) + 3 &= x^2y - 2x + \frac{1}{y} + y^2z - 2y + \frac{1}{z} + z^2x - 2x + \frac{1}{x} = \\ &= y\left(x - \frac{1}{y}\right)^2 + z\left(y - \frac{1}{z}\right)^2 + x\left(z - \frac{1}{z}\right)^2 \geq 0 \end{aligned}$$

Equality holds if and only if we have $xy = yz = zx = 1$, or, in other words, $x = y = z = 1$.

Alternative solution. It follows from $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 3$ and Cauchy-Schwarz inequality that

$$\begin{aligned} 3(x^2y + y^2z + z^2x) &= \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)(x^2y + y^2z + z^2x) \\ &= \left(\left(\frac{1}{\sqrt{y}}\right)^2 + \left(\frac{1}{\sqrt{z}}\right)^2 + \left(\frac{1}{\sqrt{x}}\right)^2\right)((x\sqrt{y})^2) + (y\sqrt{z})^2 + (z\sqrt{x})^2) \\ &\geq (x + y + z)^2. \end{aligned}$$

Therefore, $x^2y + y^2z + z^2x \geq \frac{(x + y + z)^2}{3}$ and if $x + y + z = t$ it suffices to show that $\frac{t^2}{3} \geq 2t - 3$. The latter is equivalent to $(t - 3)^2 \geq 0$. Equality holds when

$$x\sqrt{y}\sqrt{y} = y\sqrt{z}\sqrt{z} = z\sqrt{x}\sqrt{x},$$

i.e. $xy = yz = zx$ and $t = x + y + z = 3$. Hence, $x = y = z = 1$.

Comment. The inequality is true with the condition $xy + yz + zx \leq 3xyz$.

Problem 2. A *special number* is a positive integer n for which there exist positive integers a, b, c and d with

$$n = \frac{a^3 + 2b^3}{c^3 + 2d^3}.$$

Prove that:

(a) there are infinitely many special numbers;

(b) 2014 is not a special number.

Solution. (a) Every perfect cube k^3 of a positive integer is special because we can write

$$k^3 = k^3 \frac{a^3 + 2b^3}{a^3 + 2b^3} = \frac{(ka)^3 + 2(kb)^3}{a^3 + 2b^3}$$

for some positive integers a, b.

(b) Observe that $2014 = 2.19.53$. If 2014 is special, then we have,

$$x^3 + 2y^3 = 2014(u^3 + 2v^3) \tag{1}$$

for some positive integers x, y, u, v. We may assume that $x^3 + 2y^3$ is minimal with this property. Now, we will use the fact that if 19 divides $x^3 + 2y^3$, then it divides both x and y. Indeed, if 19 does not divide x, then it does not divide y too. The relation $x^3 \equiv -2y^3 \pmod{19}$ implies $(x^3)^6 \equiv (-2y^3)^6 \pmod{19}$. The latter congruence is equivalent to $x^{18} \equiv 2^6 y^{18} \pmod{19}$. Now, according to the Fermat's Little Theorem, we obtain $1 \equiv 2^6 \pmod{19}$, that is 19 divides 63, not possible.

It follows $x = 19x_1$, $y = 19y_1$, for some positive integers x_1 and y_1. Replacing in (1) we get

$$19^2(x_1^3 + 2y_1^3) = 2.53(u^3 + 2v^3) \tag{2}$$

i.e. $19 | u^3 + 2v^3$. It follows $u = 19u_1$ and $v = 19v_1$, and replacing in (2) we get

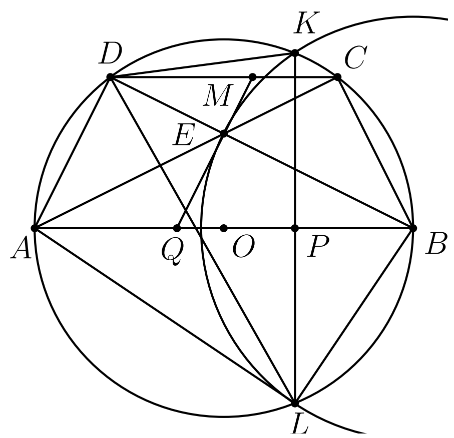
$$x_1^3 + 2y_1^3 = 2014(u_1^3 + 2v_1^3).$$

Clearly, $x_1^3 + 2y_1^3 < x^3 + 2y^3$, contradicting the minimality of $x^3 + 2y^3$.

Problem 3. Let $ABCD$ be a trapezium inscribed in a circle Γ with diameter AB. Let E be the intersection point of the diagonals AC and BD. The circle with center B and radius BE meets Γ at the points K and L, where K is on the same side of AB as C. The line perpendicular to BD at E intersects CD at M.

Prove that KM is perpendicular to DL.

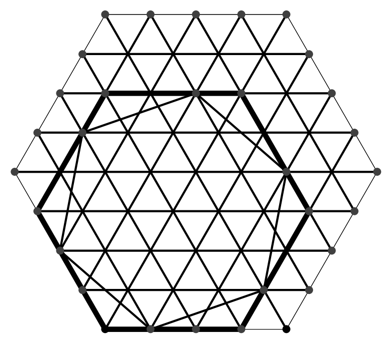
Solution. Since $AB \parallel CD$, we have that $ABCD$ is isosceles trapezium. Let O be the center of k and EM meets AB at point Q. Then, from the right angled triangle BEQ, we have $BE^2 = BO.BQ$. Since $BE = BK$, we get $BK^2 = BO.BQ$ (1). Suppose that KL meets AB at P. Then, from the right angled triangle BAK, we have $BK^2 = BP.BA$ (2)



From (1) and (2) we get $\dfrac{BP}{BQ} = \dfrac{BO}{BA} = \dfrac{1}{2}$, and therefore P is the midpoint of BQ (3). However, $DM \parallel AQ$ and $MQ \parallel AD$ (both are perpendicular to DB). Hence, $AQMD$ is parallelogram and thus $MQ = AD = BC$. We conclude that $QBCM$ is isosceles trapezium. It follows from (3) that KL is the perpendicular bisector of BQ and CM, that is, M is symmetric to C with respect to KL. Finally, we get that M is the orthocenter of the triangle DLK by using the well-known result that the reflection of the orthocenter of a triangle to every side belongs to the circumcircle of the triangle and vise versa.

Problem 4. Let n be a positive integer. A regular hexagon with side length n is divided into equilateral triangles with side length 1 by lines parallel to its sides.
Find the number of regular hexagons all of whose vertices are among the vertices of the equilateral triangles.

Solution. By a lattice hexagon we will mean a regular hexagon whose sides run along edges of the lattice. Given any regular hexagon H, we construct a lattice hexagon whose edges pass through the vertices of H, as shown in the figure, which we will call the enveloping lattice hexagon of H. Given a lattice hexagon G of side length m, the number of regular hexagons whose enveloping lattice hexagon is G is exactly m.



Yet also there are precisely $3(n-m)(n-m+1)+1$ lattice hexagons of side length m in our lattice: they are those with centres lying at most $n-m$ steps from the centre of the lattice. In particular, the total number of regular hexagons equals

$$N = \sum_{m=1}^{n} (3(n-m)(n-m+1)+1)m = (3n^2+3n)\sum_{m=1}^{n} m - 3(2m+1)\sum_{m=1}^{n} m^2 + 3\sum_{m=1}^{n} m^3.$$

Since $\sum_{m=1}^{n} m = \frac{n(n+1)}{2}$, $\sum_{m=1}^{n} m^2 = \frac{n(n+1)(2n+1)}{6}$ and $\sum_{m=1}^{n} m^3 = \left(\frac{n(n+1)}{2}\right)^2$ it is easily checked that $N = \left(\frac{n(n+1)}{2}\right)^2$.