EGMO | 2014

European Girls' Mathematical Olympiad

Antalya - Turkey

Problems and Solutions

Day 1

The EGMO 2014 Problem Committee thanks the following countries for submitting problem proposals:

- Bulgaria
- Iran
- Japan
- Luxembourg
- Netherlands
- Poland
- Romania
- Ukraine
- United Kingdom

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1. Determine all real constants t such that whenever a, b, c are the lengths of the sides of a triangle, then so are a^2+bct, b^2+cat, c^2+abt.

Proposed by S. Khan, UNK

The answer is the interval $[2/3, 2]$.

Solution 1.

If $t<2/3$, take a triangle with sides $c=b=1$ and $a=2-\epsilon$. Then $b^2+cat+c^2+abt-a^2-bct=3t-2+\epsilon(4-2t-\epsilon)\le 0$ for small positive ϵ; for instance, for any $0<\epsilon<(2-3t)/(4-2t)$.

On the other hand, if $t>2$, then take a triangle with sides $b=c=1$ and $a=\epsilon$. Then $b^2+cat+c^2+abt-a^2-bct=2-t+\epsilon(2t-\epsilon)\le 0$ for small positive ϵ; for instance, for any $0<\epsilon<(t-2)/(2t)$.

Now assume that $2/3\le t\le 2$ and $b+c>a$. Then using $(b+c)^2\ge 4bc$ we obtain

$$\begin{aligned}b^2+cat+c^2+abt-a^2-bct &= (b+c)^2+at(b+c)-(2+t)bc-a^2\\ &\ge (b+c)^2+at(b+c)-\frac{1}{4}(2+t)(b+c)^2-a^2\\ &\ge \frac{1}{4}(2-t)(b+c)^2+at(b+c)-a^2\,.\end{aligned}$$

As $2-t\ge 0$ and $t>0$, this last expression is an increasing function of $b+c$, and hence using $b+c>a$ we obtain

$$b^2+cat+c^2+abt-a^2-bct>\frac{1}{4}(2-t)a^2+ta^2-a^2=\frac{3}{4}\left(t-\frac{2}{3}\right)a^2\ge 0$$

as $t\ge 2/3$. The other two inequalities follow by symmetry.

Solution 2.

After showing that t must be in the interval $[2/3,2]$ as in **Solution 1**, we let $x=(c+a-b)/2$, $y=(a+b-c)/2$ and $z=(b+c-a)/2$ so that $a=x+y$, $b=y+z$, $c=z+x$. Then we have:

$$b^2+cat+c^2+abt-a^2-bct=(x^2+y^2-z^2+xy+xz+yz)t+2(z^2+xz+yz-xy)$$

Since this linear function of t is positive both at $t=2/3$ where

$$\frac{2}{3}(x^2+y^2-z^2+xy+xz+yz)+2(z^2+xz+yz-xy)=\frac{2}{3}((x-y)^2+4(x+y)z+2z^2)>0$$

and at $t=2$ where

$$2(x^2+y^2-z^2+xy+xz-yz)+2(z^2+xz+yz+xy)=2(x^2+y^2)+4(x+y)z>0\,,$$

it is positive on the entire interval $[2/3,2]$.

Solution 3.

After the point in **Solution 2** where we obtain

$$b^2 + cat + c^2 + abt - a^2 - bct = (x^2 + y^2 - z^2 + xy + xz + yz)t + 2(z^2 + xz + yz - xy)$$

we observe that the right hand side can be rewritten as

$$(2 - t)z^2 + (x - y)^2 t + (3t - 2)xy + z(x + y)(2 + t).$$

As the first three terms are non-negative and the last term is positive, the result follows.

Solution 4.

First we show that t must be in the interval $[2/3, 2]$ as in **Solution 1**. Then:

Case 1: If $a \geq b, c$, then $ab + ac - bc > 0$, $2(b^2 + c^2) \geq (b + c)^2 > a^2$ and $t \geq 2/3$ implies:

$$\begin{aligned} b^2 + cat + c^2 + abt - a^2 - bct &= b^2 + c^2 - a^2 + (ab + ac - bc)t \\ &\geq (b^2 + c^2 - a^2) + \frac{2}{3}(ab + ac - bc) \\ &\geq \frac{1}{3}(3b^2 + 3c^2 - 3a^2 + 2ab + 2ac - 2bc) \\ &\geq \frac{1}{3}\left[(2b^2 + 2c^2 - a^2) + (b - c)^2 + 2a(b + c - a)\right] \\ &> 0 \end{aligned}$$

Case 2: If $b \geq a, c$, then $b^2 + c^2 - a^2 > 0$. If also $ab + ac - bc \geq 0$, then $b^2 + cat + c^2 + abt - a^2 - bct > 0$. If, on the other hand, $ab + ac - bc \leq 0$, then since $t \leq 2$, we have:

$$\begin{aligned} b^2 + cat + c^2 + abt - a^2 - bct &\geq b^2 + c^2 - a^2 + 2(ab + ac - bc) \\ &\geq (b - c)^2 + a(b + c - a) + a(b + c) \\ &> 0 \end{aligned}$$

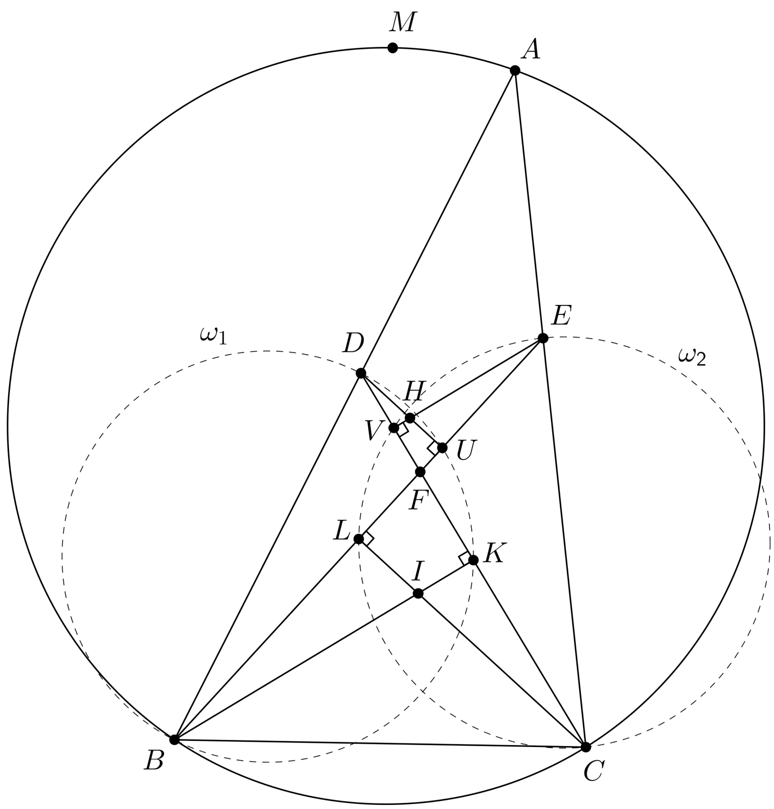
By symmetry, we are done.

2. Let D and E be two points on the sides AB and AC, respectively, of a triangle ABC, such that $DB = BC = CE$, and let F be the point of intersection of the lines CD and BE. Prove that the incenter I of the triangle ABC, the orthocenter H of the triangle DEF and the midpoint M of the arc BAC of the circumcircle of the triangle ABC are collinear.

Proposed by Danylo Khilko, UKR

Solution 1.

As $DB = BC = CE$ we have $BI \perp CD$ and $CI \perp BE$. Hence I is orthocenter of triangle BFC. Let K be the point of intersection of the lines BI and CD, and let L be the point of intersection of the lines CI and BE. Then we have the power relation $IB \cdot IK = IC \cdot IL$. Let U and V be the feet of the perpendiculars from D to EF and E to DF, respectively. Now we have the power relation $DH \cdot HU = EH \cdot HV$.



Let ω_1 and ω_2 be the circles with diameters BD and CE, respectively. From the power relations above we conclude that IH is the radical axis of the circles ω_1 and ω_2.

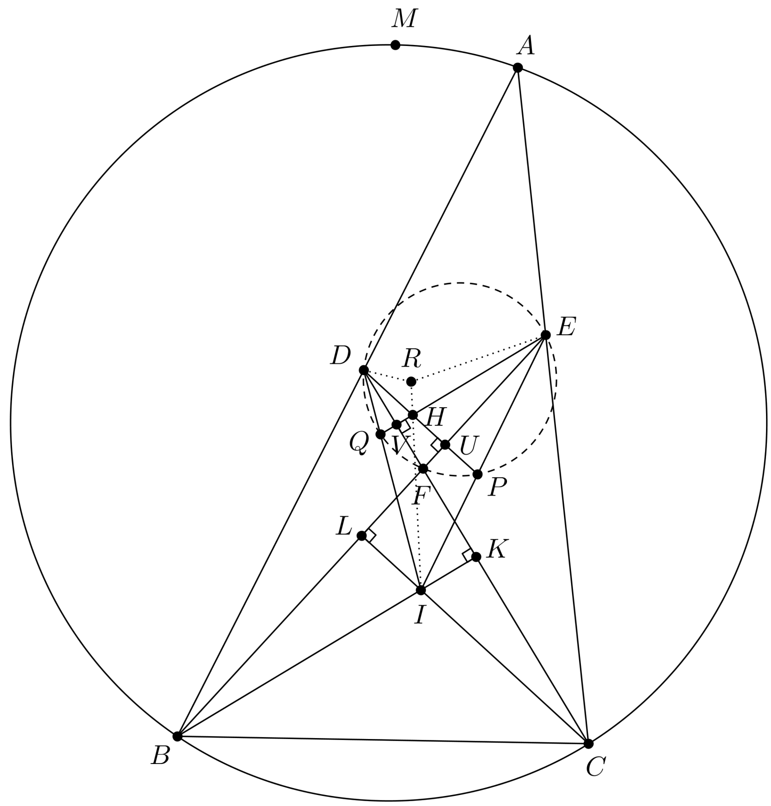
Let O_1 and O_2 be centers of ω_1 and ω_2, respectively. Then $MB = MC$, $BO_1 = CO_2$ and $\angle MBO_1 = \angle MCO_2$, and the triangles MBO_1 and MCO_2 are congruent. Hence $MO_1 = MO_2$. Since radii of ω_1 and ω_2 are equal, this implies that M lies on the radical axis of ω_1 and ω_2 and M, I, H are collinear.

Solution 2.

Let the points K, L, U, V be as in **Solution 1**. Le P be the point of intersection of DU and EI, and let Q be the point of intersection of EV and DI.

Since $DB = BC = CE$, the points CI and BI are perpendicular to BE and CD, respectively. Hence the lines BI and EV are parallel and $\angle IEB = \angle IBE = \angle UEH$. Similarly, the lines CI and DU are parallel and $\angle IDC = \angle ICD = \angle VDH$. Since $\angle UEH = \angle VDH$, the points D, Q, F, P, E are concyclic. Hence $IP \cdot IE = IQ \cdot ID$.

Let R be the second point intersection of the circumcircle of triangle HEP and the line HI. As $IH \cdot IR = IP \cdot IE = IQ \cdot ID$, the points D, Q, H, R are also concyclic. We have $\angle DQH = \angle EPH = \angle DFE = \angle BFC = 180° - \angle BIC = 90° - \angle BAC/2$. Now using the concylicity of D, Q, H, R, and E, P, H, R we obtain $\angle DRH = \angle ERH = \angle 180° - (90° - \angle BAC/2) = 90° + \angle BAC/2$. Hence R is inside the triangle DEH and $\angle DRE = 360° - \angle DRH - \angle ERH = 180° - \angle BAC$ and it follows that the points A, D, R, E are concyclic.



As $MB = MC$, $BD = CE$, $\angle MBD = \angle MCE$, the triangles MBD and MCE are congruent and $\angle MDA = \angle MEA$. Hence the points M, D, E, A are concyclic. Therefore the points M, D, R, E, A are concyclic. Now we have $\angle MRE = 180° - \angle MAE = 180° - (90° + \angle BAC/2) = 90° - \angle BAC/2$ and since $\angle ERH = 90° + \angle BAC/2$, we conclude that the points I, H, R, M are collinear.

Solution 3.

Suppose that we have a coordinate system and $(b_x, b_y), (c_x, c_y), (d_x, d_y), (e_x, e_y)$ are the coordinates of the points B, C, D, E, respectively. From $\overrightarrow{BI} \cdot \overrightarrow{CD} = 0$, $\overrightarrow{CI} \cdot \overrightarrow{BE} = 0$, $\overrightarrow{EH} \cdot \overrightarrow{CD} = 0$, $\overrightarrow{DH} \cdot \overrightarrow{BE} = 0$ we obtain $\overrightarrow{IH} \cdot (\overrightarrow{B} - \overrightarrow{C} - \overrightarrow{E} + \overrightarrow{D}) = 0$. Hence the slope of the line IH is $(c_x + e_x - b_x - d_x)/(b_y + d_y - c_y - e_y)$.

Assume that the x-axis lies along the line BC, and let $\alpha = \angle BAC$, $\beta = \angle ABC$, $\theta = \angle ACB$. Since $DB = BC = CE$, we have $c_x - b_x = BC$, $e_x - d_x = BC - BC\cos\beta - BC\cos\theta$, $b_y = c_y = 0$, $d_y - e_y = BC\sin\beta - BC\sin\theta$. Therefore the slope of IH is $(2 - \cos\beta - \cos\theta)/(\sin\beta - \sin\theta)$.

Now we will show that the slope of the line MI is the same. Let r and R be the inradius and circumradius of the triangle ABC, respectively. As $\angle BMC = \angle BAC = \alpha$ and $BM = MC$, we have

$$m_y - i_y = \frac{BC}{2}\cot\left(\frac{\alpha}{2}\right) - r \quad \text{and} \quad m_x - i_x = \frac{AC - AB}{2}$$

where (m_x, m_y) and (i_x, i_y) are the coordinates of M and I, respectively. Therefore the slope of MI is $(BC\cot(\alpha/2) - 2r)/(AC - AB)$.

Now the equality of these slopes follows using

$$\frac{BC}{\sin\alpha} = \frac{AC}{\sin\beta} = \frac{AB}{\sin\theta} = 2R,$$

hence

$$BC\cot\left(\frac{\alpha}{2}\right) = 4R\cos^2\left(\frac{\alpha}{2}\right) = 2R(1 + \cos\alpha)$$

and

$$\frac{r}{R} = \cos\alpha + \cos\beta + \cos\theta - 1$$

as

$$\frac{BC\cot(\alpha/2) - 2r}{AC - AB} = \frac{2R(1 + \cos\alpha) - 2r}{2R(\sin\beta - \sin\theta)} = \frac{2 - \cos\beta - \cos\theta}{\sin\beta - \sin\theta}$$

giving the collinearity of the points I, H, M.

Solution 4.

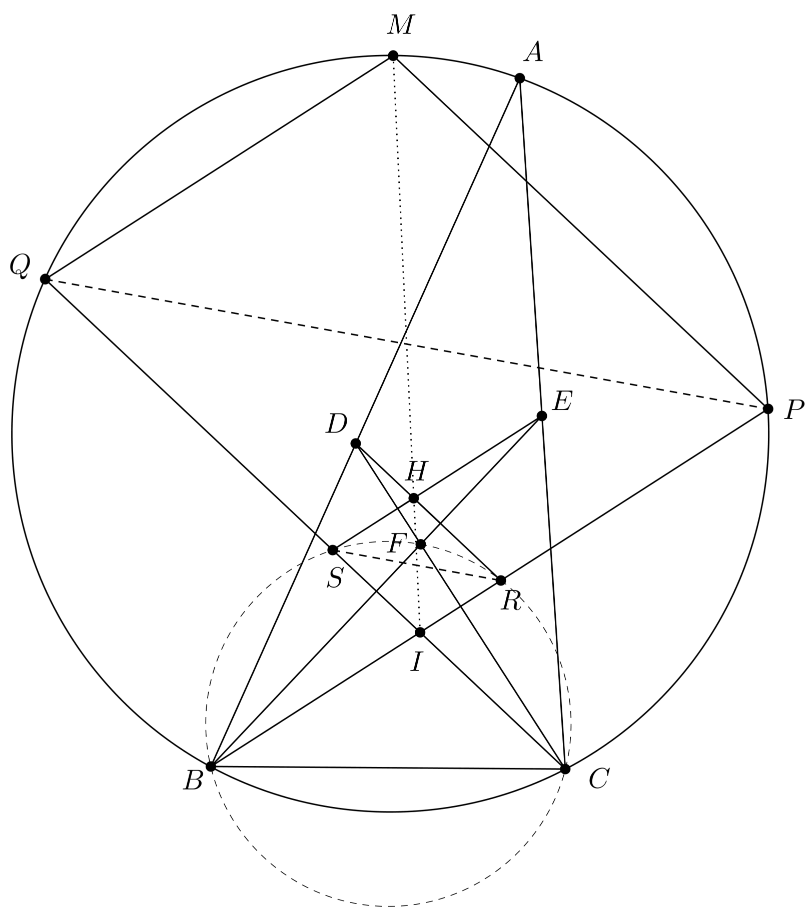
Let the bisectors BI and CI meet the circumcircle of ABC again at P and Q, respectively. Let the altitude of DEF belonging to D meet BI at R and the one belonging to E meet CI at S.

Since BI is angle bisector of the iscosceles triangle CBD, BI and CD are perpendicular. Since EH and DF are also perpendicular, HS and RI are parallel. Similarly, HR and SI are parallel, and hence $HSIR$ is a parallelogram.

On the other hand, as M is the midpoint of the arc BAC, we have $\angle MPI = \angle MPB = \angle MQC = \angle MQI$, and $\angle PIQ = (\widehat{PA} + \widehat{CB} + \widehat{AQ})/2 = (\widehat{PC} + \widehat{CB} + \widehat{BQ})/2 = \angle PMQ$. Therefore $MPIQ$ is a parallelogram.

Since CI is angle bisector of the iscosceles triangle BCE, the triangle BSE is also isosceles. Hence $\angle FBS = \angle EBS = \angle SEB = \angle HEF = \angle HDF = \angle RDF = \angle FCS$ and B, S, F, C are concyclic. Similarly, B, F, R, C are concyclic. Therefore B, S, R, C are concyclic. As B, Q, P, C are also concyclic, SR an QP are parallel.

Now it follows that $HSIR$ and $MQIP$ are homothetic parallelograms, and therefore M, H, I are collinear.



3. We denote the number of positive divisors of a positive integer m by $d(m)$ and the number of distinct prime divisors of m by $\omega(m)$. Let k be a positive integer. Prove that there exist infinitely many positive integers n such that $\omega(n) = k$ and $d(n)$ does not divide $d(a^2 + b^2)$ for any positive integers a, b satisfying $a + b = n$.

Proposed by JPN

Solution.

We will show that any number of the form $n = 2^{p-1}m$ where m is a positive integer that has exactly $k - 1$ prime factors all of which are greater than 3 and p is a prime number such that $(5/4)^{(p-1)/2} > m$ satisfies the given condition.

Suppose that a and b are positive integers such that $a + b = n$ and $d(n) \mid d(a^2 + b^2)$. Then $p \mid d(a^2 + b^2)$. Hence $a^2 + b^2 = q^{cp-1}r$ where q is a prime, c is a positive integer and r is a positive integer not divisible by q. If $q \geq 5$, then

$$2^{2p-2}m^2 = n^2 = (a + b)^2 > a^2 + b^2 = q^{cp-1}r \geq q^{p-1} \geq 5^{p-1}$$

gives a contradiction. So q is 2 or 3.

If $q = 3$, then $a^2 + b^2$ is divisible by 3 and this implies that both a and b are divisible by 3. This means $n = a + b$ is divisible by 3, a contradiction. Hence $q = 2$.

Now we have $a + b = 2^{p-1}m$ and $a^2 + b^2 = 2^{cp-1}r$. If the highest powers of 2 dividing a and b are different, then $a + b = 2^{p-1}m$ implies that the smaller one must be 2^{p-1} and this makes 2^{2p-2} the highest power of 2 dividing $a^2 + b^2 = 2^{cp-1}r$, or equivalently, $cp - 1 = 2p - 2$, which is not possible. Therefore $a = 2^t a_0$ and $b = 2^t b_0$ for some positive integer $t < p - 1$ and odd integers a_0 and b_0. Then $a_0^2 + b_0^2 = 2^{cp-1-2t}r$. The left side of this equality is congruent to 2 modulo 4, therefore $cp - 1 - 2t$ must be 1. But then $t < p - 1$ gives $(c/2)p = t + 1 < p$, which is not possible either.