32$^{\text{th}}$ BALKAN MATHEMATICAL OLYMPIAD
Athens, Hellas (May 5, 2015)

Problem 1. Let a, b and c be positive real numbers. Prove that

$$a^3b^6 + b^3c^6 + c^3a^6 + 3a^3b^3c^3 \geq abc(a^3b^3 + b^3c^3 + c^3a^3) + a^2b^2c^2(a^3 + b^3 + c^3).$$

Solution. After dividing both sides of the given inequality by $a^3b^3c^3$ it becomes

$$\left(\frac{b}{c}\right)^3 + \left(\frac{c}{a}\right)^3 + \left(\frac{a}{b}\right)^3 + 3 \geq \left(\frac{a}{c}\cdot\frac{b}{c} + \frac{b}{a}\cdot\frac{c}{a} + \frac{c}{b}\cdot\frac{a}{b}\right) + \left(\frac{a}{b}\cdot\frac{a}{c} + \frac{b}{a}\cdot\frac{b}{c} + \frac{c}{a}\cdot\frac{c}{b}\right). \tag{1}$$

Set

$$\frac{b}{a} = \frac{1}{x}, \quad \frac{c}{b} = \frac{1}{y}, \quad \frac{a}{c} = \frac{1}{z}. \tag{2}$$

Then we have that $xyz = 1$ and by substituting (2) into (1), we find that

$$x^3 + y^3 + z^3 + 3 \geq \left(\frac{y}{z} + \frac{z}{x} + \frac{x}{y}\right) + \left(\frac{x}{z} + \frac{y}{x} + \frac{z}{y}\right). \tag{3}$$

Multiplying the inequality (3) by xyz, and using the fact that $xyz = 1$, the inequality is equivalent to

$$x^3 + y^3 + z^3 + 3xyz - xy^2 - yz^2 - zx^2 - yx^2 - zy^2 - xz^2 \geq 0. \tag{4}$$

Finally, notice that by the special case of Schur's inequality

$$x^r(x-y)(x-z) + y^r(y-x)(y-z) + z^r(z-y)(z-x) \geq 0, \quad x, y, z \geq 0, r > 0,$$

with $r = 1$ there holds

$$x(x-y)(x-z) + y(y-x)(y-z) + z(z-y)(z-x) \geq 0 \tag{5}$$

which after expansion actually coincides with the congruence (4). $\square$

Remark 1. The inequality (5) immediately follows by supposing (without loss of generality) that $x \geq y \geq z$, and then writing the left hand side of the inequality (5) in the form

$$(x-y)(x(x-z) - y(y-z)) + z(y-z)(x-z),$$

which is obviously ≥ 0.

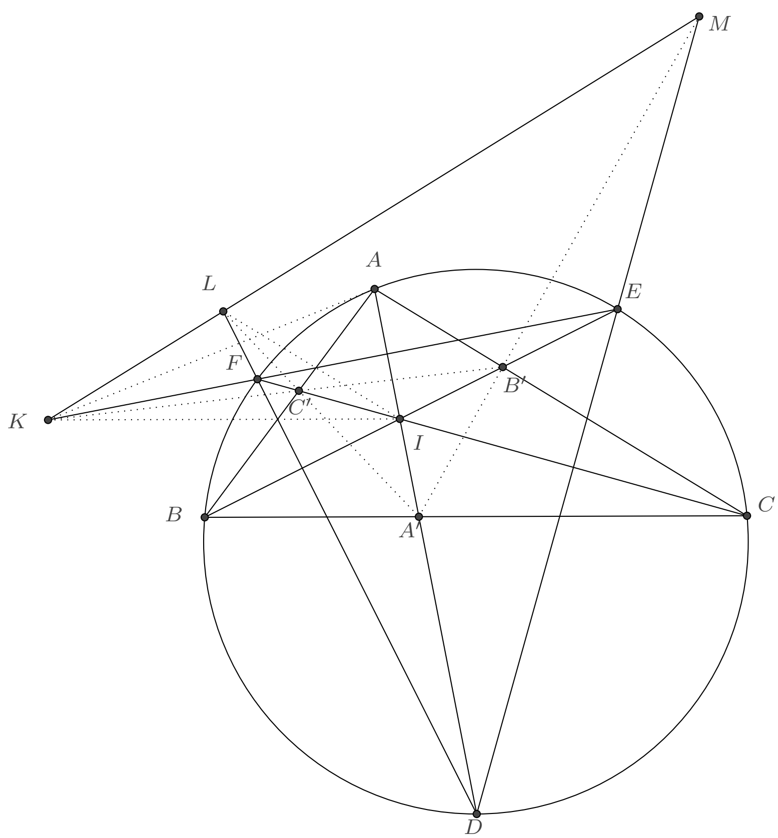
Remark 2. One can obtain the relation (4) using also the substitution $x = ab^2$, $y = bc^2$ and $z = ca^2$.

Problem 2. Let ABC be a scalene triangle with incentre I and circumcircle (ω). The lines AI, BI, CI intersect (ω) for the second time at the points D, E, F, respectively. The lines through I parallel to the sides BC, AC, AB intersect the lines EF, DF, DE at the points K, L, M, respectively. Prove that the points K, L, M are collinear.

Solution. First we will prove that KA is tangent to (ω).
Indeed, it is a well-known fact that $FA = FB = FI$ and $EA = EC = EI$, so FE is the perpendicular bisector of AI. It follows that $KA = KI$ and

$$\angle KAF = \angle KIF = \angle FCB = \angle FEB = \angle FEA,$$

so KA is tangent to (ω). Similarly we can prove that LB, MC are tangent to (ω) as well.



Let A', B', C' the intersections of AI, BI, CI with BC, CA, AB respectively. From Pascal's Theorem on the cyclic hexagon $AACDEB$ we get K, C', B' collinear. Similarly L, C', A' collinear and M, B', A' collinear.

Then from Desargues' Theorem for $\triangle DEF, \triangle A'B'C'$ which are perspective from the point I, we get that points K, L, M of the intersection of their corresponding sides are collinear as wanted. $\square$

Remark (P.S.C.). After proving that KA, LB, MC are tangent to (ω), we can argue as follows:
It readily follows that $\triangle KAF \sim \triangle KAE$ and so $\dfrac{KA}{KE} = \dfrac{KF}{KA} = \dfrac{AF}{AE}$, thus $\dfrac{KF}{KE} = \left(\dfrac{AF}{AE}\right)^2$. In a similar way we can find that $\dfrac{ME}{MD} = \left(\dfrac{CE}{CD}\right)^2$ and $\dfrac{LD}{LF} = \left(\dfrac{BD}{BF}\right)^2$. Multiplying we obtain $\dfrac{KF}{KE} \cdot \dfrac{ME}{MD} \cdot \dfrac{LD}{LF} = 1$, so by the converse of Menelaus theorem applied in the triangle DEF we get that the points K, L, M are collinear.

Problem 3. A jury of 3366 film critics are judging the Oscars. Each critic makes a single vote for his favourite actor, and a single vote for his favourite actress. It turns out that for every integer $n \in \{1, 2, \ldots, 100\}$ there is an actor or actress who has been voted for exactly n times. Show that there are two critics who voted for the same actor and for the same actress.

Solution. Let us assume that every critic votes for a different pair of actor and actress. We'll arrive at a contradiction proving the required result. Indeed:

Call the vote of each critic, i.e his choice for the pair of an actor and an actress, as a double-vote, and call as a single-vote each one of the two choices he makes, i.e. the one for an actor and the other one for an actress. In this terminology, a double-vote corresponds to two single-votes.

For each $n = 34, 35, \ldots, 100$ let us pick out one actor or one actress who has been voted by exactly n critics (i.e. appears in exactly n single-votes) and call S the set of these movie stars. Calling a, b the number of men and women in S, we have $a + b = 67$.

Now let S_1 be the set of double-votes, each having exactly one of its two corresponding single-votes in S, and let S_2 be the set of double-votes with both its single-votes in S. If s_1, s_2 are the number of elements in S_1, S_2 respectively, we have that the number of all double-votes with at least one single-vote in S is $s_1 + s_2$, whereas the number of all double-votes with both single-votes in S is $s_2 \leq ab$.

Since all double-votes are distinct, there must exist at least $s_1 + s_2$ critics. But the number of all single-votes in S is $s_1 + 2s_2 = 34 + 35 + \cdots + 100 = 4489$, and moreover $s \leq ab$. So there exist at least $s_1 + s_2 = s_1 + 2s_2 - s_2 \geq 4489 - ab$ critics.

Now notice that as $a + b = 67$, the maximum value of ab with a, b integers is obtained for $\{a, b\} = \{33, 34\}$, so $ab \leq 33 \cdot 34 = 1122$. A quick proof of this is the following: $ab = \dfrac{(a+b)^2 - (a-b)^2}{4} = \dfrac{67^2 - (a-b)^2}{4}$ which is maximized (for not equal integers a, b as $a + b = 67$) whenever $|a - b| = 1$, thus for $\{a, b\} = \{33, 34\}$.

Thus there exist at least $4489 - 1122 = 3367$ critics which is a contradiction and we are done.

$\square$

Remark. We are going here to give some motivation about the choice of number 34, used in the above solution.

Let us assume that every critic votes for a different pair of actor and actress. One can again start by picking out one actor or one actress who has been voted by exactly n critics for $n = k, k+1, \ldots, 100$. Then $a + b = 100 - k + 1 = 101 - k$ and the number of all single-votes is $s_1 + 2s_2 = k + k + 1 + \cdots + 100 = 5050 - \dfrac{k(k-1)}{2}$, so there exist at least $s_1 + s_2 = s_1 + 2s_2 - s_2 \geq 5050 - \dfrac{k(k-1)}{2} - ab$ and

$$ab = \frac{(a+b)^2 - (a-b)^2}{4} = \frac{(101-k)^2 - (a-b)^2}{4} \leq \frac{(101-k)^2 - 1}{4}.$$

After all, the number of critics is at least

$$5050 - \frac{k(k-1)}{2} - \frac{(101-k)^2 - 1}{4}.$$

In order to arrive at a contradiction we have to choose k such that

$$5050 - \frac{k(k-1)}{2} - \frac{(101-k)^2 - 1}{4} \geq 3367$$

and solving the inequality with respect to k, the only value that makes the last one true is $k = 34$.

Problem 4. Prove that among any 20 consecutive positive integers there exists an integer d such that for each positive integer n we have the inequality

$$n\sqrt{d}\,\{n\sqrt{d}\} > \frac{5}{2}$$

where $\{x\}$ denotes the fractional part of the real number x. The fractional part of a real number x is x minus the greatest integer less than or equal to x.

Solution. Among the given numbers there is a number of the form $20k + 15 = 5(4k + 3)$. We shall prove that $d = 5(4k + 3)$ satisfies the statement's condition. Since $d \equiv -1(mod\ 4)$, it follows that d is not a perfect square, and thus for any $n \in \mathbb{N}$ there exists $a \in \mathbb{N}$ such that $a + 1 > n\sqrt{d} > a$, that is, $(a + 1)^2 > n^2d > a^2$. Actually, we are going to prove that $n^2d \geq a^2 + 5$. Indeed:

It is known that each positive integer of the form $4s + 3$ has a prime divisor of the same form. Let $p \mid 4k + 3$ and $p \equiv -1(mod\ 4)$. Because of the form of p, the numbers $a^2 + 1^2$ and $a^2 + 2^2$ are not divisible by p, and since $p \mid n^2d$, it follows that $n^2d \neq a^2 + 1, a^2 + 4$. On the other hand, $5 \mid n^2d$, and since $5 \nmid a^2 + 2, a^2 + 3$, we conclude $n^2d \neq a^2 + 2, a^2 + 3$. Since $n^2d > a^2$ we must have $n^2d \geq a^2 + 5$ as claimed. Therefore,

$$n\sqrt{d}\{n\sqrt{d}\} = n\sqrt{d}(n\sqrt{d} - a) \geq a^2 + 5 - a\sqrt{a^2 + 5} > a^2 + 5 - \frac{a^2 + (a^2 + 5)}{2} = \frac{5}{2},$$

which was to be proved. $\square$